The Continuity Equation for Hybrid Systems

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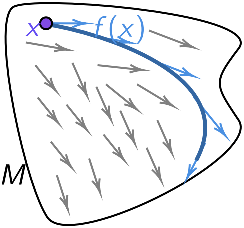
Notation

Notation	Meaning
M	n dimensional manifold
f	vector field $f : M \to TM$ with flow ϕ_t
S	the impact surface $S \subset M$, codimension 1
Δ	discrete map / the impact/reset map $\Delta : S \to M$
μ	reference volume form/reference measure on M
ρ	arbitrary measure on M
h	the density of measure ρ with respect to μ, $h : M \to \mathbb{R}$
$\mathcal{P}(M)$	all measures on M
$\Omega^n(M)$	all top/volume forms on M

Continuous vs. discrete dynamical systems

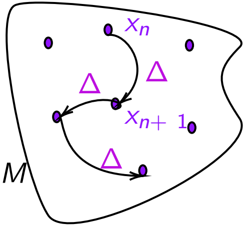
Continuous

- $\frac{d}{ds}x(s)\Big|_{s=t} = f(x(t))$
- $\phi_t(x) = x(t)$,
  $\phi : \mathbb{R} \times M \to M$



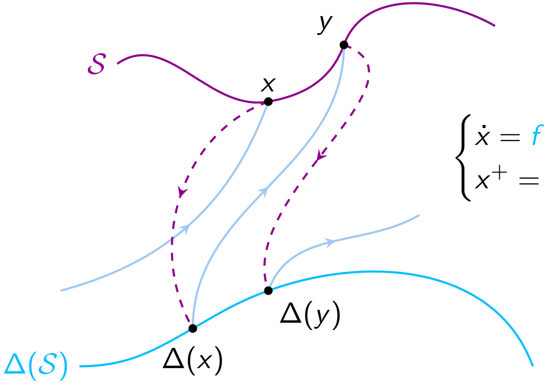
Discrete

- $x_{n+1} = \Delta(x_n), \Delta : M \to M$
- $\phi_k(x) = \Delta^k(x)$



Hybrid systems

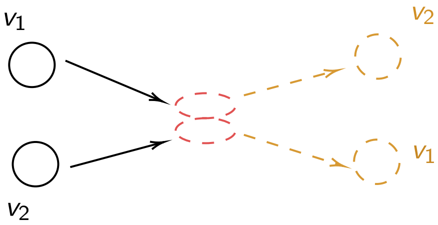
- $\mathcal{H} = (M, f, S, \Delta), \qquad f \nparallel S$



$$\begin{cases} \dot{x} = f(x), & \text{if } x \notin \mathcal{S} \\ x^{+} = \Delta(x^{-}), & \text{if } x \in \mathcal{S} \end{cases}$$

Motivation for hybrid framework

- Collisions, state transitions, discontinuities, interventions, biological phenomena, robotics



$$f = \{\dot{x} = v, \dot{v} = 0\}$$

$$S = \{x_1 = x_2\}$$

$$\Delta(x, (v_1, v_2)) = (x, (v_2, v_1))$$

Motivation for hybrid framework

- Collisions, state transitions, discontinuities, interventions, biological phenomena, robotics

src:http://www.focus.org.uk/proton neutron.php

Motivation for hybrid framework

- Collisions, state transitions[1] , discontinuities, interventions, biological phenomena, robotics

src: https://geodiode.com/biomes/savannah

[1] Scheffer M. and Carpenter S.R. "Catastrophic regime shifts in ecosystems: linking theory to observation". In: *Trends in Ecology and Evolution 18 (2003)*.

Motivation for hybrid framework

- Collisions, state transitions, discontinuities, interventions, biological phenomena, robotics

src: https://www.seafoodsource.com/

src: https://doi.org/10.3390/act11030075

Lagrangian vs Eulerian perspective

$$\text{Particles} \implies \text{densities.}$$

Lagrangian

- Precision/ accuracy
- Small number of simulations, low dimension
- Solve an ODE / SDE to obtain the trajectory

Eulerian

- Ensemble of trajectories, many simulations
- Average behaviour
- No sensitive dependence on initial conditions
- Solve a PDE to obtain the evolution of a density

[2] Saidi M. S. et al. "Comparison between Lagrangian and Eulerian approaches in predicting motion of micron-sized particles in laminar flows". In: *Atmospheric Environment* 89 (2014).

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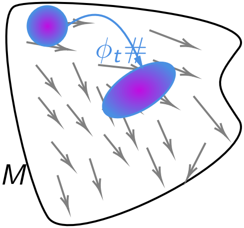
Ex: modelling aerosols though the human respiratory system[2]

[2] M. S. et al., "Comparison between Lagrangian and Eulerian approaches in predicting motion of micron-sized particles in laminar flows".

Evolution of measures

Continuous

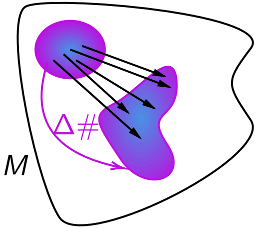
- $\phi_t(x) = x(t)$,
  $\phi : \mathbb{R} \times M \to M$



- $\rho \mapsto \phi_t\#\rho$

Discrete

- $\phi_k(x) = \Delta^k(x)$

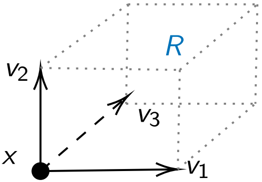


- $\rho \mapsto \Delta\#\rho$

Measures vs volume forms on a manifold

"Volume form = absolutely continuous*, infinitesimal measure"

- $\rho \in \Omega^n(M)$
- $\rho \in \mathcal{P}(M)$



- $Vol = \rho_x(v_1, v_2, v_3)$
- $\rho = h\mu$
- In coordinates: $\mu = dx_1 \dots dx_n$

- $Vol = \rho(R)$
- $\rho << \mu \implies \rho = h\mu$, $h = \frac{d\rho}{d\mu}$
- In coordinates: $\mu = Leb$

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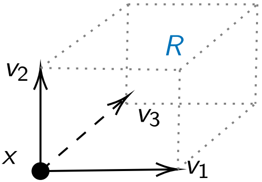

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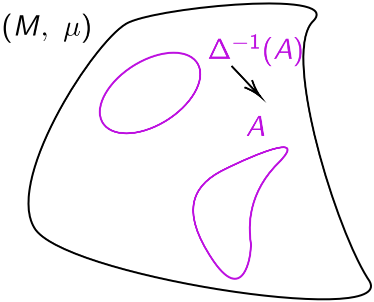
The Frobenius-Perron operator

$(M, \mathcal{B}, \mu)$ measure space and $\Delta : M \to M$ nonsingular.

Discrete[3]

The unique linear operator $P : L^1(M) \to L^1(M)$ defined by

$$\int_A Ph(x)d\mu(x) = \int_{\Delta^{-1}(A)} h(x)d\mu(x), \; \forall \; A \in \mathcal{B}$$



[3] Lasota and Mackey. *Chaos, Fractals and Noise*. Springer, 1994.

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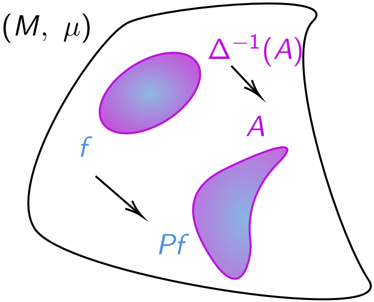

[3] Lasota and Mackey, *Chaos, Fractals and Noise*.

The Frobenius-Perron operator

$\dot{x} = f(x)$ with flow ϕ_t.

Continuous

The semigroup of transfer operators $P_t : L^1(M) \to L^1(M)$ defined by:

$$\int_A P_t(f) d\mu(x) = \int_{\phi_{-t}(A)} f(x) d\mu(x)$$

backwards in time

Motivation

- Nonlinear finite dimensional $\rightarrow$ linear infinite dimensional

[4] Lasota and Mackey, *Chaos, Fractals and Noise*.

Motivation

- Nonlinear finite dimensional $\rightarrow$ linear infinite dimensional
- Dominant eigenfunctions of P_t is the invariant densities[4]

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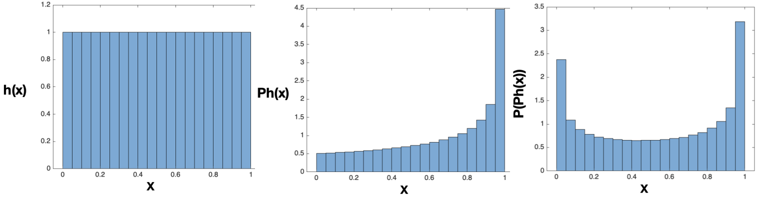
Motivation

- Nonlinear finite dimensional $\rightarrow$ linear infinite dimensional
- Dominant eigenfunctions of P_t is the invariant densities[4]
- Frobenius Perron $=$ dual of Koopman
- Study of long term behaviour.

Example

$$M = [0, 1], \quad \Delta(x) = 4x(1-x), \quad h(x) = 1$$



[4]Lasota and Mackey, *Chaos, Fractals and Noise*.

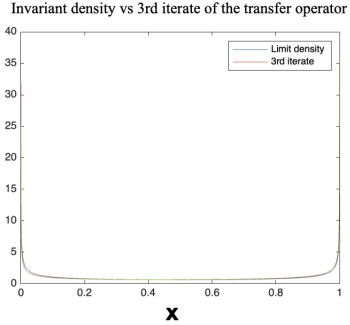
Motivation

- Eigenfunctions of P_t are invariant densities[5]
- Frobenius Perron = dual of Koopman
- Study of long term behaviour.

Example

- $M = [0, 1]$
- $\Delta(x) = 4x(1 - x)$
- $h(x) = 1$

Invariant density
$h_{inv} = \frac{1}{\pi x \sqrt{1-x}}$



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The continuity equation

Goal: Obtain an equation for the evolution of h.

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Discrete

Fix measure $\rho = h\mu$

- Let $u(k, x) = P^k h(x)$
- Change variables $\int_A Ph(x) d\mu(x) = \int_{\Delta^{-1}(A)} h(x) d\mu(x) \implies$

$$Ph(x) = \sum_{y \in \Delta^{-1}(x)} h(y) J^{-1}(y)$$

determinant of the inverse of the Jacobian matrix $(\Delta_*)_{ij} = \frac{\partial \Delta_i}{\partial x_j}$

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- $u(k,x) = \sum_{y\in\Delta^{-1}(x)} u(k-1,y)J^{-1}(y)$

The continuity equation

Continuous

- Fix initial h and define $u(t,x) = P_t h(x) =$ density of $\rho_t = \phi_t \# \rho$
- Goal: equation for u.

Lemma

If f "nice enough"[6] *then $\rho_t = \phi_t \# \rho$ satisfies the equation*

$$\frac{\partial \rho_t}{\partial t} + \nabla(f \rho_t) = 0 \tag{1}$$

in the weak sense. Conversely, any solution of (1) can be written as $\rho_t = \phi_t \# \rho$ for some flow ϕ_t.

[6] Ambrosio Luigi, Gigli Nicola, and Savaré Giuseppe. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Birkhäuser Verlag, 2005.

The continuity equation

Continuous

- Fix initial h and define $u(t,x) = P_t h(x) =$ density of $\rho_t = \phi_t \# \rho$
- Goal: equation for u.

How does the density evolve?

The continuity equation

Continuous

- Fix initial h and define $u(t,x) = P_t h(x)$ = density of $\rho_t = \phi_t \# \rho$
- Goal: equation for u.
- $\rho_t = u(t,x)\mu$ & product rule

$$\frac{\partial u}{\partial t} + du(f) + div_\mu(f)u = 0 \tag{2}$$

$\langle \nabla u, f \rangle = \mathcal{L}_f u$ = how much the density changes due to the flow.

$\mathcal{L}_f \mu$ = rate of expansion of a unit volume as it goes around the flow

The infinitesimal generator[8]

Define $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^1(M, \mu)$,

$$\mathcal{A}h =^7 \lim_{t\to 0} \frac{P_t h - h}{t}$$

Example

- $T_t h = h(x - ct) \implies \mathcal{A} = -c\frac{\partial}{\partial x}$
- Transport PDE : $\frac{\partial u}{\partial t} = \mathcal{A}u = -c\frac{\partial u}{\partial x}$

$$\text{Continuity equation} \iff A = -\mathcal{L}_f - div_\mu(f)$$

[7] in the strong sense

[8] Lasota and Mackey, *Chaos, Fractals and Noise*.

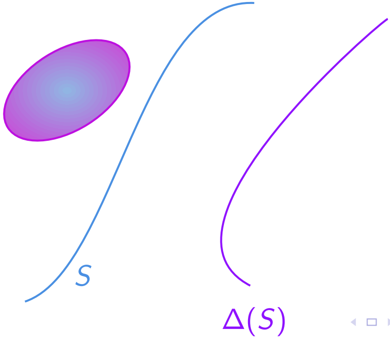
Hybrid Transfer Operators

Hybrid Frobenius Perron Operator

Definition

Let $\mathcal{H} = (M, f, S, \Delta)$ a hybrid system and let $\varphi_t^{\mathcal{H}}$ be the hybrid flow. Then the Frobenius Perron operator associated to $\mathcal{H}$ is the semigroup of operators $P_t^{\mathcal{H}} : L^1(M) \to L^1(M)$, satisfying

$$\int_A P_t^{\mathcal{H}} h(x) d\mu(x) = \int_{\varphi_{-t}^{\mathcal{H}}(A)} h(x) d\mu(x), \forall\ A \in \mathcal{B}$$



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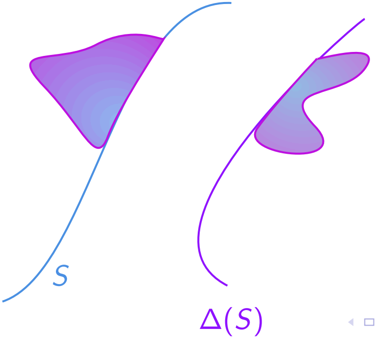

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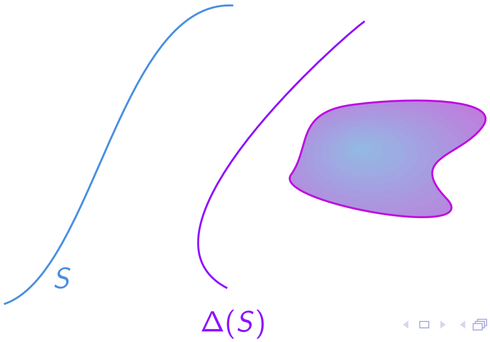

Objective

Continuity equation for the hybrid system $\iff$ infinitesimal generator of the hybrid transfer operator

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Naive approach

Discrete + continuous = hybrid

$$\begin{cases} \partial_t u(t,x) + \nabla u(t,x)\cdot f(x) = -div_\mu(f)u(t,x), & \text{if } x \notin \Delta(S) \\ u(t^+,x) = J^{-1}(\Delta^{-1}(x))u(t^-,\Delta^{-1}(x)) \text{ if } x \in \Delta(S) \end{cases}$$

after impact   before impact

Challenges

Dimensionality

- $\Delta : S \to M$, $dim(S) = n - 1$.

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- $\Delta : S \to M$, $dim(S) = n - 1$.
- Determinant of $n - 1 \times n$ matrix
- Change of variables $S = \{x_n = 0\}$

$$\implies \Delta_* = \begin{pmatrix} \partial_1 \Delta_1 & \partial_1 \Delta_2 & \dots & \partial_1 \Delta_n \\ & & \dots & \\ \partial_{n-1} \Delta_1 & \partial_{n-1} \Delta_2 & \dots & \partial_{n-1} \Delta_n \end{pmatrix}$$

Missing a row!

Challenges

Dimensionality

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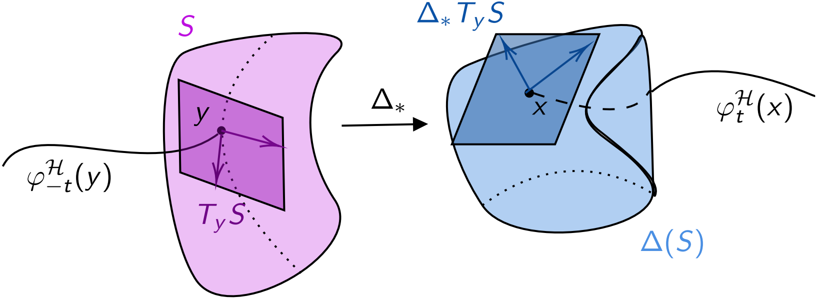
Missing a row!

Fundamental

How does infinitesimal volume change when moving through S?

Fundamental challenge

- $\mu($ n vectors$)$ = volume of hypercube
- Only $n-1$ linearly independent tangent vectors available at $y = \Delta^{-1}(x) \in S$



Requirements for the new direction

Choose: direction $\tilde{v}$ & linear map such that

- $\{\tilde{v}, T_y S\}$ span $T_y M$.
- $\det(\Delta)$ restricted to $T_y S$ and $\Delta_* T_y S = \det(\text{linear map})$

[9] Clark William and Bloch Anthony. "Invariant forms in hybrid and impact systems and a taming of Zeno". In: *Arch. Rational Mech. Anal* (2022).

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Choose: direction $\tilde{v}$ & linear map such that

- $\{\tilde{v}, T_y S\}$ span $T_y M$.
- $\det(\Delta)$ restricted to $T_y S$ and $\Delta_* T_y S = \det(\text{linear map})$

Natural choice: the flow direction and the extended differential[9]

Definition

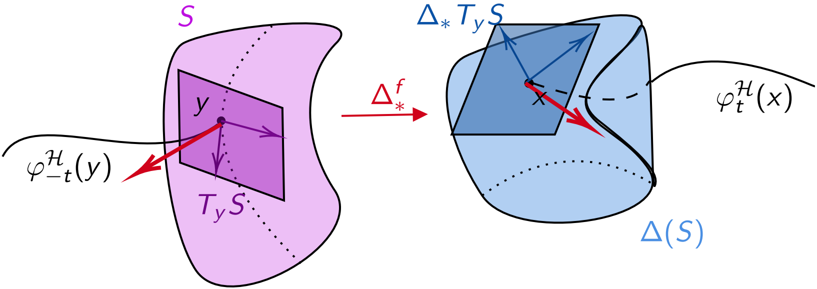
The extended differential Δ_*^f is the linear map : $T_y M \to T_x M$ defined by:

$$\begin{cases} \Delta_*^f(v) = \Delta_*(v) & \text{if } v \in T_y S \\ \Delta_*^f(v) = c \cdot f(\Delta(y)) & \text{if } v = c \cdot f(y) \in Span(f(y)) \end{cases}$$

The hybrid Jacobian: $J_\mu^f := \det(\Delta_*^f)$

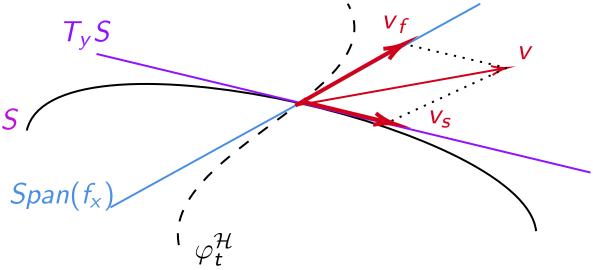
[9] William and Anthony, "Invariant forms in hybrid and impact systems and a taming of Zeno".

Illustration

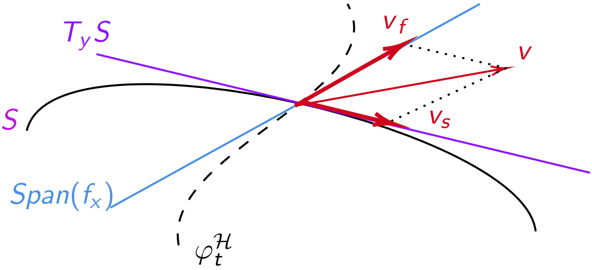


Natural decomposition of the tangent space

- $f_x \nparallel S \implies$ decomposition exists
- No extra structure needed!



Natural decomposition of the tangent space



Check:

- $J_\mu^f = \begin{vmatrix} \partial_1\Delta_1 & \dots & \partial_1\Delta_{n-1} \\ & \dots & \\ \partial_{n-1}\Delta_1 & \dots & \partial_{n-1}\Delta_{n-1} \end{vmatrix} =$

$$\begin{vmatrix} \partial_1\Delta_1 & \dots & \partial_1\Delta_{n-1} & \partial_1\Delta_n \\ & \dots & & \\ \partial_{n-1}\Delta_1 & \dots & \partial_{n-1}\Delta_{n-1} & \partial_2\Delta_n \\ f^1 & \dots & f^{n-1} & f_n \end{vmatrix} \checkmark$$

Hybrid transfer operator

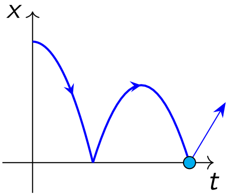
Theorem[10]
Let $\mathcal{H}$ be a smooth hybrid dynamical system and suppose $|\Delta^{-1}(\{x\})|$ is finite $\forall\ x$. Additionally, let $\mu \in \Omega^n(M)$ be a reference volume-form and suppose that $J_\mu^f \neq 0$. Then, the hybrid transfer operator $u(t,x) = P_t^{\mathcal{H}} h(x)$ satisfies the following:

$$\begin{cases} \frac{\partial u}{\partial t} + du(f) = -u\,\mathrm{div}_\mu(f) & x \notin \Delta(S) \\ u(t^+, x) = \displaystyle\sum_{y \in \Delta^{-1}(x)} \frac{1}{J_\mu^f(\Delta) \circ \Delta^{-1}}(y) u(t^-, y) & x \in \Delta(S) \end{cases}$$

[10] Et al. "A Study of the Long-Term Behavior of Hybrid Systems with Symmetries via Reduction and the Frobenius-Perron Operator". In: (2023).

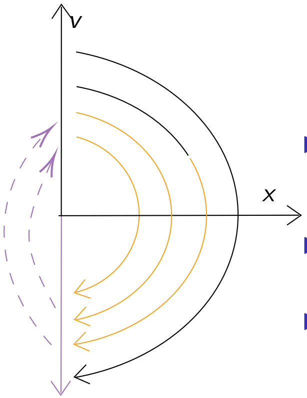
Applications

The bouncing ball



- e.o.m. $\begin{cases} \dot{x} = \frac{1}{m} v \\ \dot{v} = -mg \end{cases}$
- guard: $S = \{x = 0, v < 0\}$.
- reset: $\Delta(x, v) = (x, -c^2 v)$, $0 < c \leq 1$.

The bouncing ball



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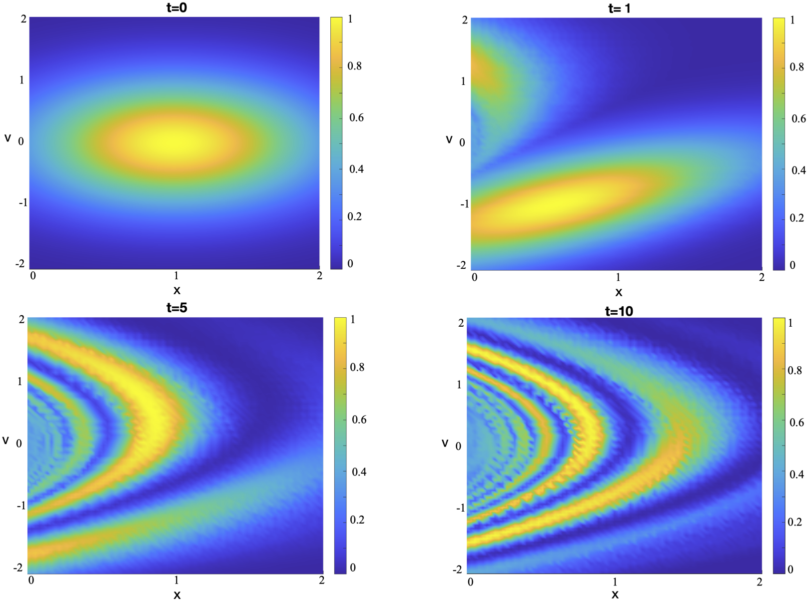
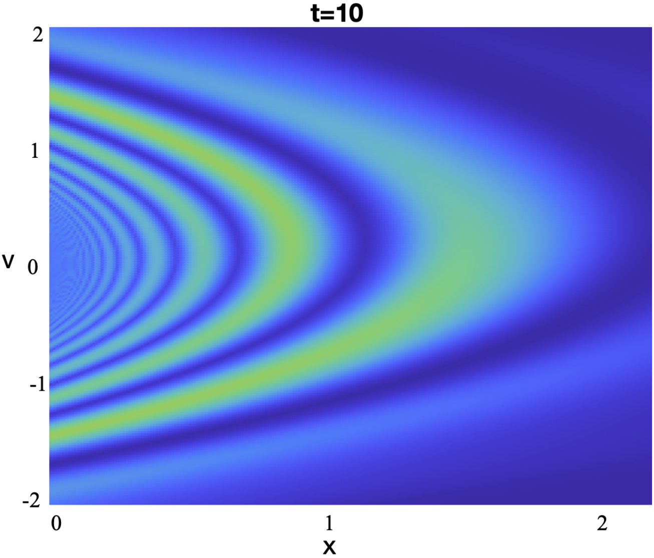
The bouncing ball results (c = 1)



Runtime $\approx 450\ s$

The bouncing ball comparison



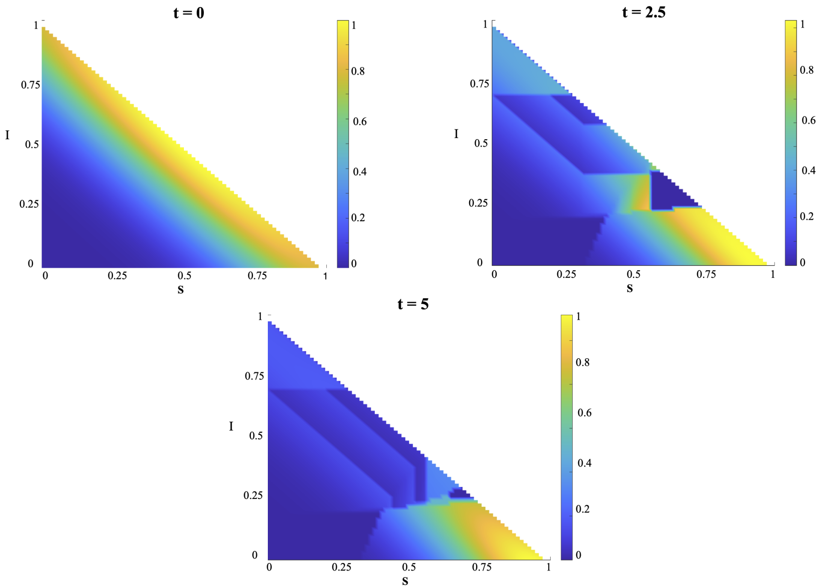
Runtime $\approx$ 3 days

An SIR model

$$\begin{cases} \dot{S} = \mu N - \dfrac{\beta S I}{N} - \mu S \\ \dot{I} = \dfrac{\beta S I}{N} - \gamma I - \mu I - \delta I \\ \dot{R} = \gamma I - \mu R \end{cases} \qquad \Delta = \begin{cases} S^+ = S^- \\ I^+ = (1-f)I^- \\ R^+ = R^- + fI^- \end{cases}$$

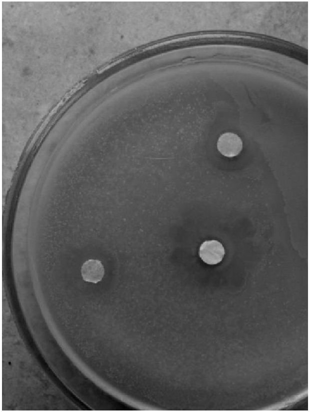
- Quarantine when threshold is reached.
- β = contact frequency, γ = recovery rate
- μ = birth/death rate, δ = mortality due to disease

SIR model results



Bacteria in competition

- Human gut, soil → enhances stability[11]
- Toxin production after threshold is reached.



src:Zimina M.I. et al. "Identification and studying of the biochemical properties of lactobacillus strains Identification and studying of the biochemical properties of lactobacillus strains". In: Life Science Journal 11.11 (January 2014), pp. 338–341

[11] Leonor García-Bayon and Laurie E. Comstock. "Bacterial antagonism in host-associated microbial communities". In: *Science* 361 (2018).

Bacteria in competition

- Human gut, soil $\rightarrow$ enhances stability[11]
- Toxin production after threshold is reached.

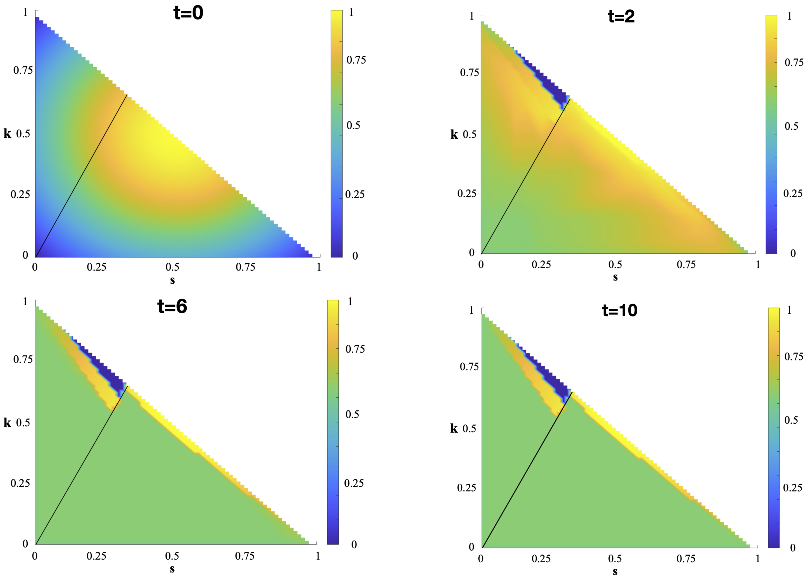
$$\dot{k} = \begin{cases} Rk\left(1 - \frac{k+s}{N}\right), & \text{if } \alpha s < k \\ (R-C)k\left(1 - \frac{k+s}{N}\right), & \text{if } s\alpha \geq k \end{cases}$$

$$\dot{s} = \begin{cases} Rs\left(1 - \frac{k+s}{N}\right), & \text{if } s\alpha < k \\ Rs\left(1 - \frac{k+s}{N}\right) - Aks, & \text{if } s\alpha \geq k \end{cases}$$

- R - growth rate, C - cost for toxin production, N - carrying capacity, A - killing rate of the toxin, α - detection threshold

[11] García-Bayon and Comstock, "Bacterial antagonism in host-associated microbial communities".

Bacteria in competition results



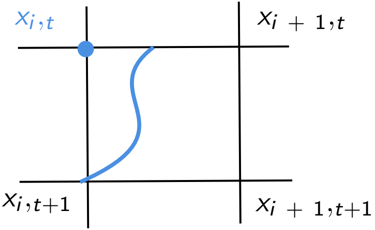
Runtime $\approx 44.78\ s$

Future research directions

Better numerical methods

Current state of the solver:

- Semi Lagrangian discretization
- Left neighbour interpolation
- Characteristics out of the grid $\implies$ run until it goes back inside the grid, or until time 0 is reached



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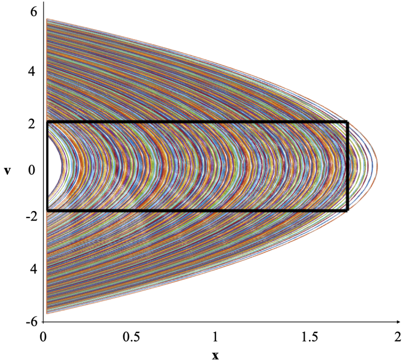

Better numerical methods

- Finite differences - spacing next to the discontinuity[12]
- Discontinuous Galerkin method[13]
- Finite volumes[14]

[12] Hogarth W. et al. "A comparative study of finite differences methods for solving the one dimensional transport equation with an initial boundary value discontinuity". In: *Computers Math. Applic* 20.11 (1990).

[13] Miloslav Feistauer Vít Dolejší. *Discontinuous Galerkin Discontinuous Galerkin Method*. Springer Series in Computational Mathematics, 2010.

[14] Aymard Benjamin et al. "A numerical method for transport equations with discontinuity flux functions: application to mathematical modeling of cell dynamics". In: *SIAM J. Sci. Comput* 36 (2013).

Introduce stochasticity

Stochastic dynamics, deterministic transition

$$\begin{cases} dX_t = f(X_t, t)dt + \sigma(X_t, t)dW_t, & \textit{if } \; X_t \notin S \\ X_t = \Delta(X_t), & \textit{if } \; X_t \in S \end{cases}$$

- Focker Plank equation instead of continuity

$$\frac{\partial}{\partial t} p + \frac{\partial}{\partial x}(\mu p) - \frac{\partial^2}{\partial x} \left(\frac{\sigma^2}{2} p \right) = 0$$

- Issue: hybrid jacobian

Example

- $f(x) = \sin(x),\ \sigma(x) = x,\ \Delta(x) = -x,\ S = \{x = \pm 0.5\}$

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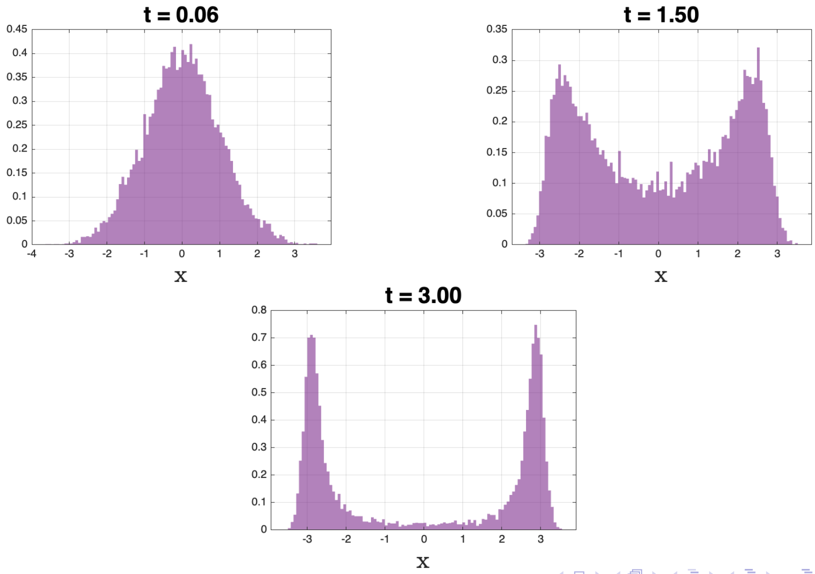

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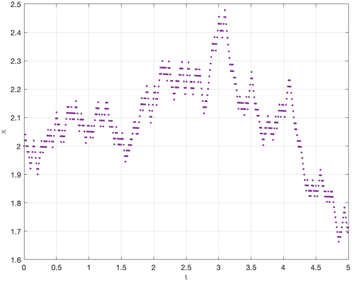

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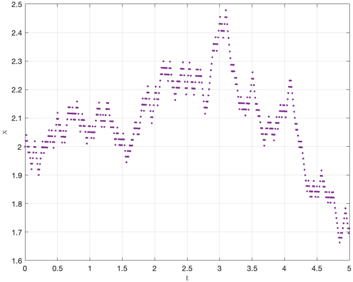


  - Teleportation after time $t \approx$Poisson(λ)
- Stochastic dynamics, stochastic transition surface

Introduce controls

$$
\begin{tikzcd}
\boxed{\begin{array}{l} \textit{Control problem} \\ \dot{x} = f(x, u) \end{array}} \arrow[r, "FP"] \arrow[d, "Cost + PMP + u^*"] & \boxed{\partial_t \rho_t + \nabla(\rho_t f^u) = 0} \arrow[d, "??"] \\
\boxed{\begin{array}{l} \textit{Hamiltonian system} \\ \dot{x} = \partial_p H \\ \dot{p} = -\partial_x H \end{array}} \arrow[r, "FP"] & \boxed{\partial_t \rho(x, p) + \nabla(\rho_t X_H) = 0}
\end{tikzcd}
$$

Thank you for your attention

Publications:

- $\sim$, Aden Shaw, Robi Huq, Kaito Iwasaki, Dora Kassabova and William Clark *A Study of the Long-Term Behavior of Hybrid Systems with Symmetries via Reduction and the Frobenius-Perron Operator* *arXiv:2309.12569 (2023)*.
- $\sim$ and William Clark *How do we walk? Using hybrid holonomy to approximate non-holonomic systems* *2022 IEEE 61st Conference on Decision and Control (CDC) 2022*.
- $\sim$, Max Ruth, Dora Kassabova, William Clark *Optimal Control of Nonholonomic Systems via Magnetic Fields* *IEEE Control Systems Letters, 7, 793-798, 2022*.

Thanks for listening

William Clark and ∼ *Optimality of Zeno Executions in Hybrid Systems*
2023 American Control Conference (ACC), 3983-3988, 2023.

∼ , Mark Walth, Robert Stephany, Gabriella Torres Nothaft, Arnaldo Rodriguez-Gonzalez, William Clark *Learning the Delay Using Neural Delay Differential Equations*
arXiv:2304.01329, 2023. poyyguyfo

William Clark, ∼ and Andrew J. Graven *A Geometric Approach to Optimal Control of Hybrid and Impulsive Systems*
arXiv:2111.11645, 2021.

William Clark and ∼ *Optimal control of hybrid systems via hybrid lagrangian submanifolds*
IFAC-PapersOnLine 54, 88-93, 2021.

Supplementary slides

Lebesque measure on a manifold

(M, g) (paracompact) Riemannian manifold

Volume form[15]

$dV_g \in \Omega^n(M)$ is the unique form such that

$$dV_g = \sqrt{det(g_{ij})} dx_1 \dots dx_n$$

Equivalently $dV_g = \epsilon^1 \wedge \cdots \wedge \epsilon^n$ for $\{\epsilon_i\}$ oriented orthonormal coframe on T^*M.

Lebesque measure

$S \subset M$ measurable if $x(S \cap U) \in \mathbb{R}^n$ measurable $\forall$ (U, x) chart.

$$\lambda_x^M(S \cap U) = \int_{x(S \cap U)} \sqrt{det(g(\partial_{x_i}, \partial_{x_j}))} d\lambda$$

[15] John M. Lee. *Introduction to Riemannian manifolds*. Springer, 2010.

Determinants[16]

M and N be n-dimensional manifolds $\mu \in \Omega^n(M)$ and $\eta \in \Omega^n(N)$. Let $F\colon TM \to TN$ be linear map. Then the determinant of F with respect to μ and η is defined to be the unique $\mathcal{C}^\infty(M)$ function such that

$$\det_{\mu\to\eta}(F)\cdot\mu = F^*\eta$$

$$\det_{\mu\to\eta}(F) = \frac{dF^*\eta}{d\mu}$$

[16] R. Abraham and J.R. Marsden. *Foundations of Mechanics*. Addison-Wesley Publishing Company, Inc., 1987.

Continuity equation precise statement[17]

Lemma

Let f be a Borel vector field satisfying

$$\int_0^T sup_B|f| + Lip(f, B)dt \leq \infty$$

$$\int_0^T \int_M |f(x)|d\mu(x)dt \leq \infty$$

and let ϕ_t be the maximal solution of $\dot{x} = f(x)$(*). Then $\rho_t = \phi_t\#\rho$ is a solution to $\partial_t\rho_t + \nabla(f\rho_t) = 0$ in the interval $(0, \tau(x) - \epsilon)\forall\ \epsilon > 0$ where $\tau(x)$ = maximal time on which solutions to (*) starting from x are defined.

[17] Luigi, Nicola, and Giuseppe, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*.

The divergence of a vector field

Let $\phi_t : M \to M$ be the flow of a vector field $f : M \to TM$. Let $\mu \in \Omega^n(M)$. Consider

$$\lim_{t \to 0} \phi_t \# \mu$$

This is a measure in $\Omega^n(M)$. Hence $\exists$ $div_\mu(f) : M \to M$ such that:

$$\lim_{t \to 0} \phi_t \# \mu = div_\mu(f)\mu$$

In coordinates with $\mu = dx_1 \wedge \cdots \wedge dx_n$, $div_\mu(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}$

The augmented differential and the hybrid Jacobian[19]

Definition

The hybrid Jacobian is the unique function $J_\mu(\Delta) : S \to \mathbb{R}$ such that

$$\Delta^* \iota^*_{\Delta(S)} i_f \alpha = J_\mu(\Delta) \iota^*_S i_f \alpha$$

$n-1$ forms on S

Theorem[18]

$\det_{\mu \to \mu} \Delta^f_* = J_\mu(f)$

[18] Et al., "A Study of the Long-Term Behavior of Hybrid Systems with Symmetries via Reduction and the Frobenius-Perron Operator".

[19] William and Anthony, "Invariant forms in hybrid and impact systems and a taming of Zeno".

Hybrid invariant differential forms[20]

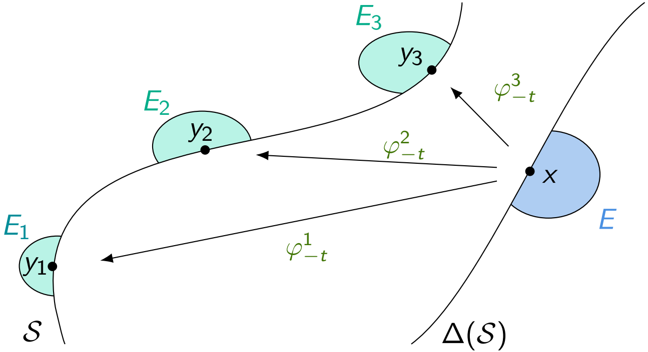
Assume $\mathcal{H}$ is a hybrid system and $\alpha \in \Omega^k(M)$.

- A differential form is invariant of $(\varphi_t^{\mathcal{H}})^*\alpha = \alpha$.
- This is equivalent to $\alpha_{\Delta(y)}(\Delta_*^f v_1, \ldots, \Delta_*^f v_n) = \alpha_y(v_1, \ldots, v_n)$
- Three conditions have to be satisfied:

$$\begin{cases} \mathcal{L}_f(\alpha) = 0 \\ \Delta^* \iota_{\Delta(S)}^* \alpha = \iota_S^* \alpha & \leftarrow \text{ specular condition} \\ \Delta^* \iota_{\Delta(S)}^* i_f \alpha = \iota_S^* i_f \alpha & \leftarrow \text{ energy condition} \end{cases}$$

[20] William and Anthony, "Invariant forms in hybrid and impact systems and a taming of Zeno".

Extension to non-invertible maps



The hybrid transfer PDE equation

The bouncing ball

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{v}{m}\frac{\partial u}{\partial x} - mg\frac{\partial u}{\partial v} = 0, & \text{for } x \neq 0; \\ u(t^+, 0, v) = u(t^-, 0, -v), & \text{otherwise.} \end{cases}$$

The SIR model

$$\frac{\partial u}{\partial t} + (\mu - \mu s - (\beta - \delta)si)\frac{\partial u}{\partial s} + (\beta si + \delta i^2 - (\gamma + \mu + \delta)i)\frac{\partial u}{\partial i}$$
$$= (2\mu - \beta(s - i) + \gamma + \delta - 3\delta i)\, u,$$
$$u(t^+, s, \alpha(1 - f)) =$$
$$= \frac{-\beta\alpha s + \delta\alpha^2 - (g + \mu + \delta)\alpha}{-\beta\alpha s(1 - f) + \delta\alpha^2(1 - f)^2 - (g + \mu + \delta)\alpha(1 - f)} u(t^-, s, \alpha).$$

The hybrid jacobian for mechanical systems

Theorem ([21])

Let $H : T^*M \to \mathbb{R}$ be a natural Hamiltonian. Let ω be the symplectic form on T^*M, and assume Δ is the impact map coming from the corner conditions. Assume moreover that S is the 0 level set of $h : M \to \mathbb{R}$.

$$\Delta = \left(x, p - (1 + c^2)\frac{p(\nabla h)}{dh(\nabla h)}dh\right)$$

Then the hybrid Jacobian is $J_{\omega^n} f = c^4$

[21] Et al., "A Study of the Long-Term Behavior of Hybrid Systems with Symmetries via Reduction and the Frobenius-Perron Operator".